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RISK PROPERTIES OF n-PERSON BARGAINING SOLUTIONS

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A new axiom for bargaining solutions on bargaining games with disposable utility is presented, called risk profit opportunity. Suppose that, in a bargaining game, one of the players (i) is replaced by a more risk averse one ($\hat{i}$). Then a solution is said to satisfy the risk sensitivity axiom if, in such a situation, it assigns (non-strictly) higher utilities to all the other players, whereas it is said to satisfy the risk profit opportunity axiom if the set of utility outcomes for the other players, restricted to the solution utility level for player $\hat{i}$, is (non-strictly) larger than the set of utility outcomes for the other players, restricted to the solution utility level of player i. For two-person bargaining solutions, risk profit opportunity is equivalent to risk sensitivity. Many n-person solutions ($n > 2$), however, satisfy the risk profit opportunity axiom but not the risk sensitivity axiom. These assertions all apply to games where all Pareto optimal outcomes are riskless. In games with risky Pareto optimal outcomes, both axioms may fail for solutions which otherwise satisfy them.

1. INTRODUCTION

Several authors have studied the effects of risk aversion in the bargaining model as introduced by Nash [9]. We mention the papers of Kihlstrom, Roth and Schmeidler [7], Roth and Rothblum [18], and Peters and Tijs [12]. Closely related are results of Kannai [5, p. 54] and Sobel [19]. All these authors consider two-person bargaining problems, and a conclusion we can draw from their results is, that, for the solutions they consider and at least in games with only riskless Pareto optimal outcomes, it is an advantage to play against the more risk averse one of two players.

In this paper, we will follow the approach similar to the one in [7] and [12]. We call a bargaining solution *risk sensitive* if,in case one of the players in a bargaining game is replaced by a more risk averse one, it assigns (non-strictly) higher utilities to the other players. In [7], Kihlstrom, Roth and Schmeidler prove that the (2-person) symmetric bargaining solutions of Nash ([9]), Kalai-Smorodinsky ([4]) and Perles-Maschler ([10]) are risk sensitive in games with only riskless Pareto optimal outcomes. Peters and Tijs [12] show that also every nonsymmetric 2-person Nash solution (cf. Harsanyi and Selten [2]) is risk sensitive. More generally, de Koster, Peters, Tijs and Wakker [8] show that every 2-person bargaining solution with the properties of Pareto optimality, individual rationality, independence of equivalent utility transformations and independence of irrelevant alternatives is risk sensitive in games with only riskless Pareto-optimal outcomes. In [13], Peters and Tijs prove that this result remains valid if we replace the property of independence of irrelevant alternatives by the individual monotonicity axiom as proposed by Kalai and Smorodinsky [4].

The results mentioned thus far, concern games with only riskless

Pareto optimal outcomes. Roth and Rothblum [18] show that in games where Pareto optimal outcomes may be risky (i.e. utility pairs achieved by non-trivial lotteries), the (symmetric 2-person) Nash solution predicts that sometimes it may be an advantage to be more risk averse, and a disadvantage to bargain against the more risk averse of two players. More precisely, if player 2 (say) is replaced by the more risk averse player $\hat{2}$, and the outcome predicted by the Nash solution in the game with player $\hat{2}$, is the utility pair of a lottery between two riskless alternatives one of which is strictly less preferred to the disagreement alternative by player $\hat{2}$, then it will be an advantage to be player $\hat{2}$, and a disadvantage to play against $\hat{2}$ instead of against the less risk averse player 2. If, however, the Nash solution predicts a utility pair achieved by a lottery between two riskless alternatives both of which are preferred by 2 to the disagreement alternative, then it will be disadvantageous to play against 2 instead of against the more risk averse player $\hat{2}$. Similar results are valid for the non-symmetric Nash solutions, partly for the Kalai-Rosenthal solution (cf. [3]), but not for the Kalai-Smorodinsky solution. See Peters and Tijs [12]).

All these results suggest that, when studying the effects of risk aversion in two-person bargaining problems, the risk sensitivity axiom seems the proper axiom to consider. Many two-person bargaining solutions proposed in literature appear to be risk sensitive. The question we consider in this paper is, whether the risk sensitivity property is also the natural property to expect for n-person bargaining solutions. In other words, should we expect that, in an n-person bargaining game, all the other players gain if one player is replaced by a more risk averse one? We feel that this is a too strong requirement. We propose a weaker axiom, called *risk profit opportunity*. This axiom does not require a solution to assign higher utilities to all other players if one of the players (say i) in a bargaining game is replaced by a more risk averse one (say $\hat{i}$), but it requires that the space of available outcomes for the other players is non-strictly larger when fixed at the solution utility level for player $\hat{i}$ than when fixed at the solution utility level for player i. We will say, below, that the *opportunity set* for the collective of players $j \neq i$ does not decrease. We will show that, for bargaining solutions restricted to games with only riskless Pareto optimal outcomes, the risk profit opportunity axiom is equivalent to the following property: If a player is replaced by a more risk averse one, then both these players prefer the alternative predicted by the solution for the less risk averse player to the alternative predicted by the solution for the more risk averse player. Whereas the risk sensitivity and risk profit opportunity axioms tell us something about the utility outcomes of the other players, this previous statement concerns the players who are being compared in risk aversion.

Except for some remarks in section 5, almost all results in this paper will concern bargaining solutions restricted to games with only riskless Pareto optimal outcomes. For these solutions, it will be shown that for $n > 2$, risk profit opportunity is strictly weaker than risk sensitivity, for $n = 2$ however these axioms are equivalent. A relation will be established between the properties of risk profit opportunity and independence of equivalent utility transformations, analogous to the relation between the latter property and risk sensitivity as established by Kihlstrom, Roth and Schmeidler [7, theorem 4] for 2-person solutions. Further, we will show that every n-person solution satisfying the independence of irrelevant alternatives axiom, has the risk profit opportunity property, but that only a small subclass of (dictatorial and almost dictatorial) solutions consists of risk sensitive solutions.

Despite the fact that there is also a class of n-person individually monotonic solutions (cf. Peters and Tijs [14]) the elements of which are risk sensitive (see section 4 below), all the mentioned facts above indicate that risk profit opportunity is the obvious extension of RS to n-person bargaining solutions restricted to games with only riskless Pareto optimal outcomes.

The organization of the paper is as follows. In section 2, we describe the formal model, define (risk) axioms for bargaining solutions, and investigate relations between these axioms. In section 3, risk behaviour is studied of n-person bargaining solutions which are independent of irrelevant alternatives. The same is done in section 4 for individually monotonic solutions. Section 5 contains some remarks with respect to risk behaviour in games with risky Pareto optimal outcomes, and section 6 concludes with a few final remarks.

2. DEFINITIONS. RELATIONS BETWEEN AXIOMS

We start with introducing the notion of a bargaining game. We need the following notation. For a compact set $S \subset \mathbb{R}^n$, let $m(S) \in \mathbb{R}^n$ be defined by

$$m_i(S) := \min\{x_i \in \mathbb{R};\ x \in S\} \text{ for all } i \in \{1,2,\ldots,n\}.$$

An *n-person bargaining game* is a pair (S,d) where

(2.1) S is a compact and convex subset of $\mathbb{R}^n$,

(2.2) $S = \{x \in \mathbb{R}^n;\ m(S) \le x \le y \text{ for some } y \in S\}$, i.e. S is equal to its *disposable hull*,

(2.3) the *disagreement outcome* d is an element of S.

Important in the sequel will be the *Pareto optimal subset*

$$P(S) := \{x \in S;\ \forall y \in S[y \geq x \Rightarrow y = x]\}.$$

The family of all n-person bargaining games is denoted by B^n.

Bargaining games correspond to bargaining situations in a way to be explained below. First, let, for some $\ell \in \mathbb{N}$, A be a non-empty compact subset of $\mathbb{R}^{\ell}$. Regard A as a *set of riskless alternatives* (for decision makers). Let L(A) denote the *set of finite lotteries* on A. A typical element ℓ of L(A) has the form

$$\ell = [p_1, a^1; p_2, a^2; \ldots; p_m, a^m] = [p_i, a^i]_{i=1}^m$$

where $m \in \mathbb{N}$, $p \in \mathbb{R}^m$ such that $p \geq 0$, $\Sigma_{i=1}^m p_i = 1$, and $a^i \in A$ for all $i = 1,2,\ldots,m$. Here ℓ denotes the lottery which results with probability p_i in alternative a^i. By identifying the riskless alternative $a \in A$ with the lottery $[1,a]$, we have $A \subset L(A)$. Elements of $L(A)\backslash A$ are called *risky alternatives.*

An *n-person bargaining situation* is an (n+2)-tuple $\langle A, \bar{a}, u^1, u^2, \ldots, u^n \rangle$ where A is as above, $\bar{a}$ is an element of A called the *disagreement alternative*, and, for every $i = 1,2,\ldots,n$, $u^i : L(A) \to \mathbb{R}$ is a function such that

(2.4) the restriction of u^i to A is continuous,

(2.5) $u^i(\ell) = \Sigma_{j=1}^m p_j\, u^i(a^j)$ for every $\ell = [p_j, a^j]_{j=1}^m \in L(A)$.

Thus (by (2.5)) u^i is a *von Neumann-Morgenstern* (NM) *utility function.* By U(A), we denote the family of all functions on L(A), satisfying (2.4) and (2.5). By BS^n, we denote the family of all n-person bargaining situations.

For every $\Gamma = \langle A, \bar{a}, u^1, u^2, \ldots, u^n \rangle$ and $\ell \in L(A)$, let $u(\ell) := (u^1(\ell), u^2(\ell), \ldots, u^n(\ell))$ and let $C_\Gamma := \mathrm{conv}(\{u(a) \in \mathbb{R}^n;\ a \in A\})$, $d_\Gamma := u(\bar{a})$. Note that $C_\Gamma = \{u(\ell);\ \ell \in L(A)\}$ in view of (2.5). We call (S_Γ, d_Γ) the *bargaining game corresponding to the bargaining situation* Γ, where

$$S_\Gamma := \{x \in \mathbb{R}^n;\ m(C_\Gamma) \le x \le y \text{ for some } y \in C_\Gamma\}$$

is the disposable hull of C_Γ. Then $(S_\Gamma,d_\Gamma) \in B^n$. Note that every $(S,d) \in B^n$ corresponds to a $\Gamma \in BS^n$ since $(S,d) = (S_\Gamma,d_\Gamma)$ if $\Gamma = \langle S,d,id^1,id^2,\ldots,id^n\rangle$ where, for every $i = 1,2,\ldots,n$, $id^i(x) := x_i$ for all $x \in S$.

Corresponding bargaining situations are important only when risk properties are studied (as will appear in the sequel). In view of this remark we call, for $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n\rangle \in BS^n$, an $x \in S_\Gamma$ a *riskless* (or *certain*) *outcome* if $x = u(a)$ for some $a \in A$, otherwise we call x a *risky outcome*. Further, we denote

$$BSC^n := \{\Gamma \in BS^n;\ \text{every } x \in P(S_\Gamma) \text{ is riskless}\}$$

where capitalized C is borrowed from the word "certain". So BSC^n is the family of n-person bargaining situations with only certain Pareto optimal outcomes in the corresponding bargaining games.

A bargaining situation $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n\rangle$ - and likewise a bargaining game (S_Γ,d_Γ) - is interpreted as a game in the following way. There are n players (bargainers) bargaining over the set $L(A)$, their preferences being represented by their NM utility functions $u^i \in U(A)$. If they cooperate and agree on an element $\ell \in L(A)$, each player i gets utility $u^i(\ell)$. If they do not cooperate, each player i gets his disagreement utility $u^i(\bar{a})$.

We now give the definitions of a bargaining map and of a bargaining solution. In the remainder of this section, Σ denotes a subfamily of B^n.

DEFINITION. A map $\phi : \Sigma \to \mathbb{R}^n$ is called an *n-person bargaining map* (on Σ) if the following axioms are satisfied:

AXIOM 1: $\phi(S,d) \ge d$ for all $(S,d) \in \Sigma$ *(Individual Rationality*, IR), AXIOM 2: $\phi(S,d) \in P(S)$ for all $(S,d) \in \Sigma$ *(Pareto Optimality*, PO).

A map $\phi : \Sigma \to \mathbb{R}^n$ is called an *n-person bargaining solution* (on Σ) if, in addition to axioms 1 and 2, it satisfies:

AXIOM 3: For every $(S,d) \in \Sigma$ and every map $T : \mathbb{R}^n \to \mathbb{R}^n$ of the form $T(x_1,x_2,\ldots,x_n) = (a_1x_1+b_1,a_2x_2+b_2,\ldots,a_nx_n+b_n)$ with $a_i,b_i \in \mathbb{R}$ and $a_i > 0$ for $i = 1,2,\ldots,n$, we have $\phi(T(S),T(d)) = T(\phi(S,d))$ if $(T(S),T(d)) \in \Sigma$ *(Independence of Equivalent Utility Transformations*, IEUT).

Since the purpose of this paper is to study risk properties, we will, forced by theorem 1(ii) below, consider only bargaining solutions in the remaining sections. Note that every bargaining map (solution) $\phi : \Sigma \to \mathbb{R}^n$ induces a map (solution) $\tilde{\phi} : \{\Gamma \in BS^n;\ (S_\Gamma,d_\Gamma) \in \Sigma\} \to \mathbb{R}^n$ for bargaining situations, in a natural way, i.e. $\tilde{\phi}(\Gamma) := \phi(S_\Gamma,d_\Gamma)$ for every $\Gamma \in BS^n$ with $(S_\Gamma,d_\Gamma) \in \Sigma$. Instead of $\tilde{\phi}$ we write ϕ.

The following definition induces a partial ordering on $U(A)$ with respect to risk aversion.

DEFINITION. Let $u,v \in U(A)$ where A is a compact subset of $\mathbb{R}^\ell$. We call (a player with utility function) v *more risk averse* than (a player with utility function) u (notation: v MR u) if there exists a continuous increasing concave function $k : \mathrm{conv}(u(A)) \to \mathbb{R}$ such that $v(a) = k \circ u(a)$ for all $a \in A$.

This measure of risk aversion is already implicit in the work of Arrow [1] and Pratt [15], and Kihlstrom and Mirman [6]. For more details, see our paper [12] where we use a modification of the approach proposed by Yaari [21].

Let $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n\rangle \in BS^n$, let $i \in \{1,2,\ldots,n\}$, and let $C_i(\Gamma)$ denote the family of all continuous increasing concave functions $k : \mathrm{conv}(u^i(A)) \to \mathbb{R}$. Let, for $k_i \in C_i(\Gamma)$,

$K^i(\Gamma) := \langle A,\bar{a},u^1,\ldots,u^{i-1},k_i \circ u^i,u^{i+1},\ldots,u^n \rangle$. Then $K^i(\Gamma)$ is the element of BS^n derived from Γ by replacing player i with utility function u^i by the more risk averse player with utility function $k_i \circ u^i$. Note that only the values $k_i(u^i(a))$ for $a \in A$ are important, since then $k_i \circ u^i$ is determined on $L(A)$ by (2.5).

In the following, Y denotes a subset of BS^n which is closed under replacing players by more risk averse ones, and $\phi : Y \to \mathbb{R}^n$ is a bargaining map. For $i \in \{1,2,\ldots,n\}$, the map $\pi_{-i} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ assigns to a vector $x \in \mathbb{R}^n$ the vector $\pi_{-i}(x) = (x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ which is obtained from x by deleting the i-th coordinate. Let $\Gamma \in Y$. Then the *opportunity set* $O_{-i}(\Gamma,\phi)$ for the bargainers $j \neq i$ with respect to Γ and ϕ, is defined by

$$O_{-i}(\Gamma,\phi) := \pi_{-i}\{x \in S_\Gamma;\ x_i = \phi_i(S_\Gamma,d_\Gamma)\}.$$

The opportunity set $O_{-i}(\Gamma,\phi)$ consists of those utility (n-1)-tuples, available for the collective $\{j \in \{1,2,\ldots,n\};\ j \neq i\}$, if bargainer i receives $\phi_i(S_\Gamma,d_\Gamma)$.

We are now sufficiently equipped to define the following axioms concerning risk.

DEFINITION. We say that the bargaining map $\phi : Y \to \mathbb{R}^n$ is *risk sensitive* (RS) if

AXIOM 4 For every $\Gamma \in Y$, every $i \in \{1,2,\ldots,n\}$ and every $k_i \in C_i(\Gamma)$, we have

(RS_i) $\quad \phi_j(K^i(\Gamma)) \geq \phi_j(\Gamma)$ for every $j \in \{1,2,\ldots,i-1,i+1,\ldots,n\}$.

We say that ϕ has the *risk profit opportunity* property (RPO) if

AXIOM 5 For every $\Gamma \in Y$, every $i \in \{1,2,\ldots,n\}$ and every $k_i \in C_i(\Gamma)$, we have

(RPO_i) $\quad O_{-i}(\Gamma,\phi) \subset O_{-i}(K^i(\Gamma),\phi)$.

We say that ϕ is *equality consistent* (EC) if

AXIOM 6 For every $\Gamma \in Y$, every $i \in \{1,2,\ldots,n\}$ and every $k_i \in C_i(\Gamma)$, we have

$$(EC_i) \qquad \text{if } O_{-i}(\Gamma,\phi) = O_{-i}(K^i(\Gamma),\phi), \text{ then } \pi_{-i}(\phi(\Gamma)) = \pi_{-i}(\phi(K^i(\Gamma))).$$

Axiom 4, which was introduced by Kihlstrom, Roth and Schmeidler in [7], states that, if a player is replaced by a more risk averse one, the solution assigns (non-strictly) higher utilities to the opposing players. Axiom 5 states that, under the same conditions, the opportunity set for the opposing players with respect to ϕ, does not decrease. Axiom 6 states that, under the same conditions again, if the opportunity set for the opposing players with respect to ϕ does not change, then their solution payoffs do not change either. Axiom 6 could have been stated in a more general, and may be more natural, fashion; we will, however, use it only in connection with RPO.

The larger part of this paper will concern bargaining games with only riskless Pareto optimal outcomes, or, equivalently, bargaining situations in BSC^n. In the following lemma, we collect some facts w.r.t. applying increasing concave transformations on the utility functions of the players in a bargaining situation in BSC^n. We first introduce some additional notations, and a definition.

For $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n\rangle \in BS^n$, let $N := \{1,2,\ldots,n\}$ denote the set of n players. For $j \in N$, $k_j \in C_j(\Gamma)$ and $x \in \mathbb{R}^n$ such that $x_j \in \text{conv}(u^j(A))$, let

$$K^j(x_1,\ldots,x_j,\ldots,x_n) := (x_1,\ldots,x_{j-1},k_j(x_j),x_{j+1},\ldots,x_n)$$

and

$$K^j(T) := \{K^j(y);\ y \in T\}$$

where $T \subset \mathbb{R}^n$ such that $y_j \in \text{conv}(u^j(A))$ for all $y \in T$.

Further, the *utopia point* $u(S_\Gamma,d_\Gamma) = (u_1(S_\Gamma,d_\Gamma),u_2(S_\Gamma,d_\Gamma),\ldots,u_n(S_\Gamma,d_\Gamma))$

is defined by

$$u_\ell(S_\Gamma,d_\Gamma) := \max\{x_\ell;\ x \in S_\Gamma,\ x \geq d_\Gamma\} \text{ for all } \ell \in N.$$

LEMMA 1. Let $\Gamma = \langle A,\bar{a},u^1,u^2,\dots,u^n\rangle \in BSC^n$, $j \in N$, $k_j \in C_j(\Gamma)$, $z \in P(S_\Gamma)$. Then we have:

(i) $d_{K^j(\Gamma)} = K^j(d_\Gamma)$, (ii) $u(S_{K^j(\Gamma)},d_{K^j(\Gamma)}) = K^j(u(S_\Gamma,d_\Gamma))$,

(iii) $P(S_{K^j(\Gamma)}) = K^j(P(S_\Gamma))$,

(iv) $\pi_{-j}\{x \in S_{K^j(\Gamma)};\ x_j = k_j(z_j)\} = \pi_{-j}\{x \in S_\Gamma;\ x_j = z_j\}$,

(v) for all $x,y \in S_\Gamma$ and all $i \in N$ such that $x_i \leq y_i$, we have

$$\pi_{-i}\{s \in S_\Gamma;\ s_i = y_i\} \subset \pi_{-i}\{s \in S_\Gamma;\ s_i = x_i\}.$$

PROOF. All facts follow elementarily from the definitions. (v) is a consequence of *(among other facts)* (2.2). □

Note that only (i) and (v) of Lemma 1 also hold for games with risky Pareto optimal outcomes, hence for any $\Gamma \in BS^n$. The main result of this section is theorem 1, the proof of which depends on some lemmas.

LEMMA 2. Let $\phi : BSC^n \to \mathbb{R}^n$ be a bargaining map. Let ϕ be risk sensitive. Then ϕ has the risk profit opportunity property and is equality consistent.

PROOF. Let $\Gamma = \langle A,\bar{a},u^1,u^2,\dots,u^n\rangle \in BSC^n$, $j \in N$, $k_j \in C_j(\Gamma)$. Then, by lemma 1(iii), we have

(2.6) $\quad K^j(\Gamma) \in BSC^n$.

We first wish to prove (RPO_j). By PO and lemma 1(iii), there exists an element $z \in P(S_\Gamma)$ such that $\phi(K^j(\Gamma)) = K^j(z)$. By ($RS_j$), we have

(2.7) $\quad \phi_i(K^j(\Gamma)) = z_i \geq \phi_i(\Gamma)$ for all $i \in N\backslash\{j\}$.

By (2.7) and PO, we have

(2.8) $\quad z_j \leq \phi_j(\Gamma)$.

By (2.8) and lemma 1(v), we obtain

(2.9) $\quad \pi_{-j}\{x \in S_\Gamma;\ x_j = \phi_j(\Gamma)\} \subset \pi_{-j}\{x \in S_\Gamma;\ x_j = z_j\}$.

By (2.9) and lemma 1(iv), we have

(RPO$_j$) $\quad O_{-j}(\Gamma,\phi) \subset O_{-j}(K^j(\Gamma),\phi)$.

Secondly, if $O_{-j}(\Gamma,\phi) = O_{-j}(K^j(\Gamma),\phi)$, then (2.7) and PO imply that $\pi_{-j}(\phi(\Gamma)) = \pi_{-j}(\phi(K^j(\Gamma)))$, which proves (EC$_j$). □

LEMMA 3. Let $\phi : BSC^n \to \mathbb{R}^n$ be a bargaining map, let $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n\rangle \in BSC^n$, and let, for every $i \in N$, $k_i \in C_i(\Gamma)$ be an affine function. Then we have

(i) if ϕ has the risk profit opportunity property, then $k_i(\phi_i(\Gamma)) = \phi_i(K^i(\Gamma))$ for every $i \in N$,

(ii) if ϕ has the risk profit opportunity property and ϕ is equality consistent, then ϕ is independent of equivalent utility transformations.

PROOF. (i) Assume, ϕ satisfies RPO. Since, for every $i \in N$, k_i as well as its inverse map are in $C_i(\Gamma)$, we have, by applying (RPO$_i$) twice,

(2.10) $\quad O_{-i}(\Gamma,\phi) = O_{-i}(K^i(\Gamma),\phi)$, for every $i \in N$.

From (2.10), PO and lemma 1(iii), it follows that $k_i(\phi_i(\Gamma)) = \phi_i(K^i(\Gamma))$ for all $i \in N$.

(ii) Assume, in addition, that ϕ satisfies EC. From (i), (2.10), and (EC$_1$), we have $K^1(\phi(\Gamma)) = \phi(K^1(\Gamma))$. Repeating this argument (n-1) more times, we find:

$$K^n \circ K^{n-1} \circ\ldots\circ K^1(\phi(\Gamma)) = \phi(K^n \circ K^{n-1} \circ\ldots\circ K^1(\Gamma)),$$

from which we conclude that ϕ satisfies IEUT, since k_i is (positive) affine for every $i \in N$. □

THEOREM 1. Let $\phi : BS^n \to \mathbb{R}^n$ be a bargaining map. Then:

(i) On BSC^n, if ϕ is risk sensitive, then ϕ has the risk profit opportunity property and is equality consistent.

(ii) If ϕ satisfies risk profit opportunity and equality consistency on BSC^n, then ϕ is independent of equivalent utility transformations on BS^n.

(iii) If $n = 2$, then, on BSC^2, ϕ is risk sensitive iff ϕ has the risk profit opportunity property.

PROOF. (i) Lemma 2.

(ii) By lemma 3(ii), if ϕ satisfies RPO and EC on BSC^n, then ϕ satisfies IEUT on BSC^n. Let $\Gamma \in BS^n$, and suppose ϕ satisfies IEUT on BSC^n. $(S_\Gamma, d_\Gamma) = (S_{\Gamma'}, d_{\Gamma'})$ where $\Gamma' := \langle S_\Gamma, d_\Gamma, id^1, id^2, \ldots, id^n \rangle$ is an element of BSC^n, hence $\phi(T(S_\Gamma), T(d_\Gamma)) = \phi(T(S_{\Gamma'}), T(d_{\Gamma'})) = T(\phi(S_{\Gamma'}, d_{\Gamma'})) = T(\phi(S_\Gamma, d_\Gamma))$ for every positive affine transformation T as in the definition of axiom 3. So ϕ satisfies IEUT on BS^n.

(iii) On BSC^2, the implication RS $\Rightarrow$ RPO follows from lemma 2, the implication RPO $\Rightarrow$ RS follows immediately from PO. □

The solutions considered in section 3, show that, for $n > 2$, risk sensitivity is a strictly stronger axiom than risk profit opportunity. Example 1 shows that, for $n > 2$, RPO does not imply EC (for $n = 2$, EC is a trivial axiom in view of PO). Theorem 1 (ii) is an n-person analogon of a theorem in Kihlstrom, Roth and Schmeidler [7, theorem 4] which says that every bargaining map $\phi : BSC^2 \to \mathbb{R}^2$ satisfies IEUT if it satisfies RS. Forced by the implication in theorem 1(ii), we will only study bargaining solutions (i.e. bargaining maps satisfying IEUT) in the following sections.

Note that, in theorem 1, all conditions are given on BSC^n. A reason for this will be provided by theorem 3 below. First we give an important alternative characterization of the RPO axiom.

THEOREM 2. Let $\phi : BSC^n \to \mathbb{R}^n$ be a bargaining map. Then the following two assertions are equivalent:

(i) ϕ satisfies RPO.

(ii) For every $\Gamma = \langle A, \bar{a}, u^1, u^2, \ldots, u^n \rangle \in BSC^n$, for every $j \in N$ and $k_j \in C_j(\Gamma)$, we have

$$u^j(a) \geq u^j(b),\ k_j \circ u^j(a) \geq k_j \circ u^j(b)$$

where $a,b \in A$ such that $(u^1(a),\dots,u^j(a),\dots,u^n(a)) = \phi(\Gamma)$ and $(u^1(b),\dots,u^{j-1}(b),\ k_j \circ u^j(b),\ u^{j+1}(b),\dots,u^n(b)) = \phi(K^j(\Gamma))$.

PROOF. Suppose (ii) holds. With notations and conditions as in (ii), we have $u(b) \in P(S_\Gamma)$ and $(u^1(a),\dots,u^{j-1}(a),\ k_j \circ u^j(a),\ u^{j+1}(a),\dots,u^n(a)) \in K^j(P(S_\Gamma))$, in view of lemma 1(iii). Then, since $u^j(a) \geq u^j(b)$, we have, in view of lemma 1(v),

(2.11) $\quad O_{-j}(\Gamma,\phi) \subset \pi_{-j}\{x \in S_\Gamma;\ x_j = u^j(b)\}$

and also, in view of lemma 1(iv),

(2.12) $\quad \pi_{-j}\{x \in S_\Gamma;\ x_j = u^j(b)\} = O_{-j}(K^j(\Gamma),\phi)$.

Combining (2.11) and (2.12) gives (RPO_j), and hence RPO, since the argument holds for every $j \in N$.

Now suppose (i) holds. Assume notations and conditions as in (ii). Since $O_{-j}(\Gamma,\phi) \subset O_{-j}(K^j(\Gamma),\phi)$, there is an element c in A such that $\pi_{-j}(u(c)) \geq \pi_{-j}(u(a))$ and $k_j \circ u^j(c) = k_j \circ u^j(b)$. By PO, $u^j(c) \leq u^j(a)$, hence $k_j \circ u^j(b) \leq k_j \circ u^j(a)$, and then also $u^j(b) \leq u^j(a)$. So (ii) holds.□

Theorem 2 says that, if in a bargaining situation one of the players is replaced by a more risk averse one, then an alternative assigned by an RPO bargaining map in the game with the less risk averse player is preferred by both players (the less risk averse one and the more risk averse one) to an alternative assigned by that map in the game with the more risk averse player. For 2-person bargaining games with also risky Pareto optimal outcomes, kindlike results can be derived. A remark made by Roth and Rothblum [18, p. 642] already points in that direction.

The following theorem provides a reason why most results in this paper are restricted to bargaining games with only riskless Pareto optimal outcomes.

THEOREM 3. Let $\phi : BS^n \to \mathbb{R}^n$ be a bargaining map. Then ϕ satisfies neither of the axioms RS and RPO.

PROOF. Let $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n\rangle \in BS^n$, where $A = \{\bar{a},a^1,a^2,\ldots,a^n\}$ and $u^i \in U(A)$ are defined by

(2.13) $\quad u^i(\bar{a}) = 0,\ u^i(a^j) = -1$ if $i \neq j$, $u^i(a^i) = n$, for all $i,j = 1,2,\ldots,n.$

Let, for all $i = 1,2,\ldots,n$, the vectors x^i and e^i in $\mathbb{R}^n$ be defined by

(2.14) $\quad x^i_j = u^j(a^i),\ e^i_j = 1$ if $j = i$, $e^i_j = 0$ if $j \neq i$, for all $j = 1,2,\ldots,n.$

Then

(2.15) $\quad d_\Gamma = 0$, S_Γ is the disposable hull of $\mathrm{conv}\{0,x^i;\ i=1,2,\ldots,n\}$,

$P(S_\Gamma) = \mathrm{conv}\{x^i;\ i = 1,2,\ldots,n\}$,

$\{x \in P(S_\Gamma);\ x \geq d_\Gamma = 0\} = \mathrm{conv}\{e^i;\ i = 1,2,\ldots,n\}$.

Let, for every $\beta \in (\frac{n-1}{2}, n]$, the increasing concave function $k_\beta : \mathbb{R} \to \mathbb{R}$ be defined by

(2.16) $\quad k_\beta(t) = t$ for all $t \in (-\infty,0]$, $k_\beta(n) = \beta$, k_β is linear on $[0,\infty)$.

Let, for $x \in \mathbb{R}^n$ and $\beta \in (\frac{n-1}{2}, n]$, $x^{(\beta)} := (x_1,x_2,\ldots,x_{n-1},k_\beta(x_n))$, and $S^\beta := \{x^{(\beta)} \in \mathbb{R}^n;\ x \in S_\Gamma\}$. Then $d_\Gamma^{(\beta)} = d_\Gamma = 0$ for all $\beta \in (\frac{n-1}{2}, n]$. Note that $(S^n,d_\Gamma^{(n)}) = (S_\Gamma,d_\Gamma)$. Also note that the set $\{x \in P(S^\beta);\ x \geq 0\}$ shrinks to $\{0\}$ if β tends to $\frac{n-1}{2}$. Therefore, if we suppose for a moment that ϕ satisfies RS or RPO, we must have that $\phi(S_\Gamma,d_\Gamma) = e^n$. By the same argument, we have that $\phi(S_\Gamma,d_\Gamma) = e^i$ for all $i \in \{1,2,\ldots,n\}$, a contradiction. So ϕ satisfies neither RS nor RPO. $\square$

The following example shows that RPO does not imply EC.

EXAMPLE 1. The bargaining map $\phi : BS^3 \to \mathbb{R}^3$ is defined as follows. Let $\Gamma \in BS^3$. Then $\phi(S_\Gamma,d_\Gamma) := z$ such that $z_1 := \max\{x_1;\ x \in P(S_\Gamma),\ x \geq d_\Gamma\}$, and (z_2,z_3) is such that

$$(z_2-(d_\Gamma)_2)^t(z_3-(d_\Gamma)_3)^{1-t} = \max\{(x_2-(d_\Gamma)_2)^t(x_3-(d_\Gamma)_3)^{1-t};x\in P(S_\Gamma),x\geq d_\Gamma,x_1=z_1\}$$

where $t \in [0,1)$ is such that $t(1-t)^{-1} = z_1^2$.

Then, on BSC^3, ϕ satisfies RPO but not EC. If player 1 is replaced by a more risk averse player, then the opportunity set O_{-1} for players 2 and 3 does not change, so ϕ satisfies (RPO_1); generally, however, the parameter t changes, so ϕ does not satisfy (EC_1), hence not EC. To verify that ϕ satisfies (RPO_2) and (RPO_3), hence RPO, we refer to section 3, theorem 4. Also notice, that ϕ does not satisfy axiom 3, IEUT.

3. RISK PROPERTIES OF IIA-SOLUTIONS

In this section, we consider bargaining solutions $\phi : BS^n \to \mathbb{R}^n$ with the following property which was introduced by Nash [9].

AXIOM 7. For all (S,d), $(T,e) \in B^n$ with $d = e$, $S \subset T$ and $\phi(T,e) \in S$, we have $\phi(S,d) = \phi(T,e)$ (*Independence of Irrelevant Alternatives*, IIA).

There have been many discussions on this axiom in literature, see e.g., Raiffa [16], or Kalai and Smorodinsky [4], so we will not add another one. Here is our main result.

THEOREM 4. Let $\phi : BS^n \to \mathbb{R}^n$ be a bargaining solution satisfying IIA. Then ϕ satisfies RPO and EC on BSC^n.

PROOF. Let $\Gamma \in BSC^n$, $j \in N$, and $k_j \in C_j(\Gamma)$. Put $z := \phi(\Gamma)$, $\hat{z} := \phi(K^j(\Gamma))$. Since ϕ satisfies IEUT, we may suppose without loss of generality that

(3.1) $\quad K^j(d_\Gamma) = d_\Gamma$.

We wish to prove

(3.2) $\quad O_{-j}(\Gamma,\phi) \subset O_{-j}(K^j(\Gamma),\phi)$, and

(3.3) $\quad \pi_{-j}(z) = \pi_{-j}(\hat{z})$ if $O_{-j}(\Gamma,\phi) = O_{-j}(K^j(\Gamma),\phi)$.

First suppose that $\hat{z}_j = (d_\Gamma)_j$. Then, since $z_j \geq (d_\Gamma)_j$ by IR, we have in view of (3.1) and lemma 1(iv) and (v),

(3.4) $\quad O_{-j}(\Gamma,\phi) \subset O_{-j}(K^j(\Gamma),\phi)$.

If there is equality in (3.4), then let $V := \{x \in S_\Gamma;\ x_j = (d_\Gamma)_j\}$. Then $z \in V$, $\hat{z} \in V$, $V \subset S_\Gamma$, $V \subset S_{K^j(\Gamma)}$, hence, by IIA, $\phi(V,d_\Gamma) = z = \hat{z}$. So we have proved (3.2) and (3.3) for the case $\hat{z}_j = (d_\Gamma)_j$.

Now suppose that $\hat{z}_j > (d_\Gamma)_j$. In view of lemma 1(iii) there is a unique point y in $P(S_\Gamma)$ such that $\pi_{-j}(y) = \pi_{-j}(\hat{z})$. Since ϕ satisfies IEUT, we may normalize, in addition to (3.1), in such a way that

(3.5) $\quad (\hat{z} =)\ K^j(y) = y.$

The concavity of the function k_j, (3.1) and (3.5) then imply

(3.6) $\quad k(\lambda) \geq \lambda$ for all $\lambda \in [(d_\Gamma)_j, y_j]$, $k(\lambda) \leq \lambda$ for all $\lambda \geq y_j$.

Suppose that $\hat{z}_j > z_j$ ($\geq (d_\Gamma)_j$). Let

$$T := \{x \in \mathbb{R}^n;\ d_\Gamma \leq x \leq s \text{ for some } s \in \operatorname{conv}\{d_\Gamma, z, \hat{z}\}\}.$$

Then $(T,d_\Gamma) \in B^n$, and $T \subset S_\Gamma$ since $\hat{z} \in S_\Gamma$ by (3.5), $T \subset S_{K^j(\Gamma)}$ since $z \in S_{K^j(\Gamma)}$ by (3.6). So, by IIA, $\phi(T,d_\Gamma) = z = \hat{z}$, which contradicts our assumption $\hat{z}_j > z_j$. Hence, $\hat{z}_j \leq z_j$, and then, by (3.5) and lemma 1(v), we conclude that (3.2) holds. If there is equality in (3.2), then $z_j = \hat{z}_j$, and just as before, $\phi(T,d_\Gamma) = z = \hat{z}$. So also (3.3) holds for this case.□

The remainder of this section will be devoted to a more detailed study of risk properties of solutions belonging to a family of IIA-solutions. First we introduce some notations and definitions, in order to describe that family. Let $N = \{1,2,\ldots,n\}$ again denote the player set in an n-person bargaining game (or situation).

DEFINITION. A *weighted ordered partition of* N is an object H of the form $H = \langle N^1,\omega^1,N^2,\omega^2,\ldots,N^\ell,\omega^\ell\rangle$, where $(N^1,N^2,\ldots,N^\ell)$ is an ordered partition of N (without empty elements), and, for every $i = 1,2,\ldots,\ell$, $\omega^i \in \mathbb{R}^n$ such that $\omega^i_j = 0$ if $j \notin N^i$, $\omega^i_j > 0$ if $j \in N^i$, and $\Sigma_{j=1}^n \omega^i_j = 1$. N^i is called the *i-th class of* H. H^N denotes the family of all weighted ordered partitions of N.

To every $H \in H^N$, $H = \langle N^1, \omega^1, N^2, \omega^2, \ldots, N^\ell, \omega^\ell \rangle$ we associate an n-person bargaining solution ϕ^H as follows. Let, for $(S,d) \in B^n$,

$$S^1 := \arg\max\{\Pi_{i \in \overline{N}^1} (x_i - d_i)^{\omega_i^1};\ x \in P(S),\ x \geq d\}, \text{ where}$$

$$\overline{N}^1 := \{i \in N^1; \text{there is an } x \in S \text{ such that } x_i > d_i\}.$$

Let, for $j = 2,3,\ldots,\ell$,

$$S^j := \arg\max\{\Pi_{i \in \overline{N}^j} (x_i - d_i)^{\omega_i^j};\ x \in S^{j-1}\}, \text{ where}$$

$$\overline{N}^j := \{i \in N^j; \text{ there is an } x \in S^{j-1} \text{ such that } x_i > d_i\}.$$

Then S^ℓ consists of exactly one point, say z, and we define $\phi^H(S,d) := z$. The map $\phi^H : B^n \to \mathbb{R}^n$ is an n-person bargaining solution, i.e. it satisfies axioms 1-3, and it also satisfies IIA. Actually, one can prove that for every n-person bargaining solution ϕ which satisfies IIA and an additional axiom (called *degeneracy consistency*), there is an $H \in H$ such that $\phi = \phi^H$. This additional axiom is a more general form of equality consistency. It requires ϕ to be consistent in assigning outcomes to non-degenerate and degenerate games, where we call a game (or a subset of outcomes) degenerate for player i if there is no outcome which i strictly prefers to the disagreement outcome. For proofs, more details, and discussions, see Peters [11]. In this paper, we are interested in investigating risk properties for solutions belonging to $\{\phi^H;\ H \in H^N\}$.

If $N = \{1,2\}$, and $H = \langle N, (\frac{1}{2},\frac{1}{2}) \rangle$, then ϕ^H is the (2-person symmetric) Nash solution (see Nash [9]). If $H = \langle N, (t, 1-t) \rangle$ for some $t \in (0,1)$, then ϕ^H is a nonsymmetric Nash solution, see Harsanyi and Selten [2]. If $H = \langle \{1\}, (1,0), \{2\}, (0,1) \rangle$ $(\langle \{2\}, (0,1), \{1\}, (1,0) \rangle)$ then ϕ^H is the dictator solution for player 1 (2), see de Koster et al. [8]. We have enumerated all 2-person IIA-solutions, since, in the two-person case, degeneracy consistency is guaranteed by Pareto optimality.

There do exist solutions ϕ^H which are risk sensitive on BSC^n. This is the content of the following proposition. As before, for $i \in N$, e^i is the vector in $\mathbb{R}^n$ with i-th coordinate equal to 1 and all other coordinates equal to 0.

PROPOSITION 1. Let $H \in \mathcal{H}^N$. Then:

(i) If $H = \langle\{\pi(1)\},e^{\pi(1)},\{\pi(2)\},e^{\pi(2)},\ldots,\{\pi(n)\},e^{\pi(n)}\rangle$ for some permutation $\pi : N \to N$, then ϕ^H is risk sensitive on BSC^n.

(ii) If $H = \langle\{\pi(1)\},e^{\pi(1)},\{\pi(2)\},e^{\pi(2)},\ldots,\{\pi(n-2)\},e^{\pi(n-2)},\{\pi(n-1),\pi(n)\},$ $\alpha e^{\pi(n-1)}+(1-\alpha)e^{\pi(n)}\rangle$ for some permutation $\pi : N \to N$ and $\alpha \in (0,1)$, then ϕ^H is risk sensitive on BSC^n.

PROOF. Let $\Gamma \in BSC^n$.

(i) Let π be a permutation of N, and let H be as in (i) above. If a player, say $\pi(i)$ for $i \in N$, is replaced by a *more risk averse* player, then, by definition of ϕ^H, the solution outcome changes only (possibly) for player $\pi(i)$. So ϕ^H is risk sensitive on BSC^n.

(ii) Let π be a permutation of N, and let H be as in (ii) above. If a player $\pi(i)$, $i < n-1$, is replaced by a more risk averse player, then the solution outcome predicted by ϕ^H changes only (possibly) for player $\pi(i)$. If player $\pi(n-1)$ ($\pi(n)$) is replaced by a more risk averse player, then the solution outcome does not change for all players $\pi(j)$, $j < n-1$. In view of this, RPO (cf. theorem 4) and PO, the solution outcome for player $\pi(n)$ ($\pi(n-1)$) may change only to his advantage. So ϕ^H is risk sensitive on BSC^n. □

In particular, proposition 1 implies that every 2-person solution ϕ^H is risk sensitive on BSC^2. This result also follows from theorems 4 and 1(iii).

Not every solution ϕ^H is risk sensitive on BSC^n. This will be shown by the following example.

EXAMPLE 2. Let $\Gamma \in BSC^3$ such that $S_\Gamma = S$ and $d_\Gamma = d$, where S and d are defined by

$$S := \mathrm{conv}\{(0,0,0),(1,0,0),(1,0,1),(0,0,1),(0,1,0)\},\ d := (0,0,0).$$

Then

$$P(S) = \mathrm{conv}\{(0,1,0),(1,0,1)\}.$$

Let $k_3 \in C_3(\Gamma)$ be defined by $k(\lambda) = \sqrt{\lambda}$ for all $\lambda \in [0,1]$. Note that

$$S_{K^3(\Gamma)} = K^3(S),\ d_{K^3(\Gamma)} = K^3(d) = 0\ (= (0,0,0)).$$

Then

$$P(K^3(S)) = K^3(P(S)) = \{(\alpha,1-\alpha,\sqrt{\alpha}) \in \mathbb{R}^3;\ \alpha \in [0,1]\}.$$

(i) Let $H \in H^N$, $H = <\{1,2,3\},\omega>$, hence $\omega > 0$. Straightforward calculations show:

$$\phi^H(S,d) = (\omega_1+\omega_3,\omega_2,\omega_1+\omega_3),$$

$$\phi^H(K^3(S),K^3(d)) = ((2\omega_1+\omega_3)(2-\omega_3)^{-1},(2-2\omega_1-2\omega_3)(2-\omega_3)^{-1},$$
$$(2\omega_1+\omega_3)^{\frac{1}{2}}(2-\omega_3)^{-\frac{1}{2}}).$$

So $\phi_1^H(S,d) > \phi_1^H(K^3(S),K^3(d))$ which implies that ϕ is not risk sensitive.

(ii) Let $H \in H^N$, $H = <\{2,3\},\omega,\{1\},(1,0,0)>$, hence $\omega = (0,\omega_2,\omega_3) \geq 0$ with $\omega_2,\omega_3 \neq 0$ and $\omega_2+\omega_3 = 1$. Again, straightforward calculations show:

$$\phi^H(S,d) = (\omega_3,\omega_2,\omega_3),$$

$$\phi^H(K^3(S),K^3(d)) = (\omega_3(1+\omega_2)^{-1},2\omega_2(1+\omega_2)^{-1},\omega_3^{\frac{1}{2}}(1+\omega_2)^{-\frac{1}{2}}.$$

So $\phi_1^H(S,d) > \phi_1^H(K^3(S),K^3(d))$ which implies that ϕ is not risk sensitive.

We use example 2 to show that the converse of proposition 1 also holds. More specifically, we have the following theorem.

THEOREM 5. Let $H \in H^N$, then ϕ^H is risk sensitive iff H is as in proposition 1(i) or (ii), on BSC^n.

PROOF. In view of proposition 1, we still have to prove: if H is not as in proposition 1(i) or (ii), then ϕ^H is not risk sensitive. Let $H \in H^N$,

$H = \langle N^1,\omega^1,N^2,\omega^2,\dots,N^\ell,\omega^\ell\rangle$ where $\ell \le n-1$. W.l.o.g. we may suppose: if $i \in N^h$, $j \in N^m$, then $i < j$ for all $i,j \in N$ and $1 \le h < m \le \ell$. The assumption that H is not as in proposition 1 implies that either

case 1: there exists $j \in N$ and $h \in \{1,2,\dots,\ell\}$ such that $j,j+1,j+2 \in N^h$,

or

case 2: there exists $j \in N$ and $h \in \{1,2,\dots,\ell-1\}$ such that $j,j+1 \in N^h$ and $j+2 \in N^{h+1}$.

We first consider case 1. So let H be as in case 1. Let $(S,0) \in B^3$ be as in example 2. Let $(T,0) \in B^n$ be defined by

$$T := \{x \in \mathbb{R}^n;\ (x_j,x_{j+1},x_{j+2}) \in S,\ x_i \in [0,1] \text{ if } i \notin \{j,j+1,j+2\}\}.$$

Then $P(T) = \{x \in T;\ (x_j,x_{j+1},x_{j+2}) \in P(S),\ x_i = 1 \text{ if } i \notin \{j,j+1,j+2\}\}$, and $\phi_i^H(T,0) = 1$ if $i \notin \{j,j+1,j+2\}$, $\phi_j^H(T,0) = (\omega_j^h+\omega_{j+2}^h)\sigma^{-1}$, $\phi_j^H(K^{j+2}(T),0) = (2\omega_j^h+\omega_{j+2}^h)(2\sigma-\omega_{j+2}^h)^{-1}$, where k is as in example 2, and $\sigma := \omega_j^h + \omega_{j+1}^h + \omega_{j+2}^h$. Since $\omega_j^h,\omega_{j+1}^h,\omega_{j+2}^h > 0$, we have $\phi_j^H(T,0) > \phi_j^H(K^{j+2}(T),0)$, hence ϕ^H is not risk sensitive.

Next, if H is as in case 2, then apply a similar argument as in case 1, now using part (ii) of example 2, and letting the players j,j+1,j+2 play the roles of players 2,3,1, respectively, in example 2. This also leads to the conclusion that ϕ^H is not risk sensitive. □

Theorem 5 shows that only a relatively small subclass of $\{\phi^H;\ H \in \mathcal{H}^N\}$ consists of risk sensitive solutions. This subclass consists of n! dictatorial solutions and a family of almost-dictatorial solutions each one determined by a number in (0,1) and an ordered partition of N out of ½n! possible ones, as proposition 1 shows.

4. RISK PROPERTIES OF IM-SOLUTIONS

The *individual monotonicity* (IM) *axiom* was introduced by Kalai and

Smorodinsky [4]. They characterized by it the (2-person, symmetric) Raiffa solution (Raiffa [16]). The n-person version of this axiom is inconsistent with axioms 1-3 for bargaining solutions defined on the whole class B^n of n-person bargaining games (Roth [17]). In Peters and Tijs [14], however, a family of n-person IM-solutions is characterized, with the aid of monotonic curves, where these solutions are defined on a subclass B^n_* of B^n. This subclass B^n_* consists of exactly those bargaining games $(S,d) \in B^n$ such that, for every $x \in S$ and $j \in N$, we have: if $x \geq d$, $x \notin P(S)$ and $x_j < u_j(S,d)$, then there exists an $\varepsilon > 0$ such that $x+\varepsilon e^j \in S$. We note here that B^n_* is closed under taking continuous increasing concave transformations of the utility axes.

We restrict ourselves in this section to bargaining situations of which the corresponding bargaining games are in B^n_* and have all Pareto optimal outcomes riskless, i.e. we restrict ourselves to

$$BSC^n_* := \{\Gamma \in BSC^n;\ (S_\Gamma, d_\Gamma) \in B^n_*\}.$$

We first state the IM-axiom.

AXIOM 8. For every $i \in N$ and all $(S,d),(T,e) \in B^n_*$ with $S \subset T$, $d = e$ and $u_j(S,d) = u_j(T,e)$ for every $j \in N\backslash\{i\}$, we have $\phi_i(S,d) \leq \phi_i(T,e)$.

Kalai and Smorodinsky [4] criticized Nash's IIA-axiom, and as an alternative, they formulated (the 2-person version of) axiom 8.

The following theorem states that IM implies RS.

THEOREM 6. Let $\phi : BSC^n_* \to \mathbb{R}^n$ be an individually monotonic bargaining solution. Then ϕ is risk sensitive.

PROOF. Let $\Gamma \in BSC^n_*$ and $k_i \in C_i(\Gamma)$. We wish to prove:

(4.1) $\quad \phi_j(K^i(\Gamma)) \geq \phi_j(\Gamma)$ for all $j \in N\backslash\{i\}$.

In view of IEUT, we may normalize $S_{K^i(\Gamma)}$ in such a way that

$$(4.2) \qquad (d_\Gamma)_i = (d_{K^i(\Gamma)})_i,\ u_i(S_\Gamma,d_\Gamma) = u_i(S_{K^i(\Gamma)},d_{K^i(\Gamma)}).$$

In view of lemma 1(i) and (ii), (4.2) implies that

$$(4.3) \qquad k_i((d_\Gamma)_i) = (d_\Gamma)_i,\ k_i(u_i(S_\Gamma,d_\Gamma)) = u_i(S_\Gamma,d_\Gamma) \text{ and}$$
$$d_\Gamma = d_{K^i(\Gamma)},\ u(S_\Gamma,d_\Gamma) = u(S_{K^i(\Gamma)},d_{K^i(\Gamma)}).$$

From IR, PO and IM, it follows that

$$(4.4) \qquad \phi(S_\Gamma,d_\Gamma) = \phi(\{x \in S_\Gamma;\ x \geq d_\Gamma\},d_\Gamma)$$

and

$$(4.5) \qquad \phi(S_{K^i(\Gamma)},d_{K^i(\Gamma)}) = \phi(\{x \in S_{K^i(\Gamma)};\ x \geq d_{K^i(\Gamma)}\},d_{K^i(\Gamma)}).$$

From (4.3) and the concavity of k_i, it follows that

$$(4.6) \qquad k_i(\lambda) \geq \lambda \text{ for all } \lambda \in [(d_\Gamma)_i,\ u_i(S_\Gamma,d_\Gamma)].$$

Lemma 1(iii) and (4.6) imply

$$(4.7) \qquad \{x \in S_\Gamma;\ x \geq d_\Gamma\} \subset \{x \in S_{K^i(\Gamma)};\ x \geq d_{K^i(\Gamma)}\}.$$

Now (4.7), (4.3)-(4.5) and IM imply (4.1). □

The result of theorem 6 can be explained as follows. For bargaining solutions (satisfying axioms 1-3) - purely mathematically seen - risk sensitivity can be considered as a special case of (individual) monotonicity. The difference is, roughly, that in the case of RS only continuous increasing concave transformations of the utility axes are allowed, whereas in the case of IM many more transformations can be applied.

5. SOME REMARKS ON GAMES WITH RISKY PARETO OPTIMAL OUTCOMES

Contrary to the many results known in literature with respect to risk behavior of bargaining solutions in games with riskless Pareto optimal outcomes, not much attention is paid yet to games in which there may be risky Pareto optimal outcomes, i.e. outcomes which can be achieved only by drawing lotteries between riskless (certain) alternatives. The results mentioned in the third paragraph of section 1, i.e. the results obtained by Roth and Rothblum [18], and Peters and Tijs [12], are all inexhaustive, that is, they do not cover the whole family BS^2. For individually monotonic solutions as described in section 4, not much can be said on their risk properties in games corresponding to bargaining situations in $BS^n \backslash BSC^n$, since these solutions depend on the ideal point of a game which may be determined by risky outcomes and therefore vary quite unpredictably if a player is replaced by a more risk averse one. See [12] for an example concerning the (2-person, symmetric) Kalai-Smorodinsky solution. More can be said about risk properties of 2-person IIA-solutions (see [18], [12]) in games with risky Pareto optimal outcomes, due to the fact that the disagreement outcome is taken to be riskless in the considered family of bargaining situations, but these results too are not exhaustive. Summarizing, the main reason for the indicated lack of results is simply the fact that not many regularities *can* be obtained. In this respect, see also theorem 3. Another reason is also the mathematical tediousness involved in the eventual proofs, as a price to be paid for very incomplete results.

Here, we restrict ourselves to two examples. The first example indicates an analogon, in terms of underlying alternatives in the bargaining situation, of the results known for 2-person IIA-solutions. The second example shows that theorem 2 (the characterization of RPO in terms of alternatives instead of utilities) does not hold anymore in games with risky Pareto optimal outcomes.

EXAMPLE 3. Let $\Gamma = \langle A,\bar{a},u^1,u^2,u^3\rangle \in BS^3$ be such that $A = \{\bar{a},a^1,a^2,a^3,a^4\}$, $u(\bar{a}) = (0,0,0)$, $u^3(a^1) < 0 < u^3(a^2) = u^3(a^3) < u^3(a^4)$, $u^j(a^i) > 0$ for $j \in \{1,2\}$ and $i \in \{1,2,3,4\}$. Let $\phi : BS^3 \to \mathbb{R}^3$ be an IIA-solution, and let $k_3 \in C_3(\Gamma)$. Let $\ell, \hat{\ell} \in L(A)$ be such that $u(\ell) = \phi(\Gamma)$, $\hat{u}(\hat{\ell}) = \phi(K^3(\Gamma))$, where $\hat{u} := (u^1, u^2, \hat{u}^3)$ and $\hat{u}^3 \in U(A)$, $\hat{u}^3(a) = k_3(u^3(a))$ *for* all $a \in A$.

Then we have:

(i) Let $\ell = [p_1, a^2;\ p_2, a^3;\ p_3, a^4]$ with $p_i \geq 0$ for all $i = 1,2,3$ and $\Sigma_{i=1}^{3} p_i = 1$. Then $u^3(\ell) \geq u^3(\hat{\ell})$, $\hat{u}^3(\ell) \geq \hat{u}^3(\hat{\ell})$.

(ii) Let $\ell = [p_1, a^1;\ p_2, a^2;\ p_3, a^3]$ with $p_1 > 0$, $p_2, p_3 \geq 0$ and $\Sigma_{i=1}^{3} p_i = 1$. Then $u^3(\ell) \leq u^3(\hat{\ell})$, $\hat{u}^3(\ell) \leq \hat{u}^3(\hat{\ell})$.

Proof. (i) In view of IEUT, we may normalize $S_{K^3(\Gamma)}$ in such a way that

(5.1) $$d_{K^3(\Gamma)} = d_\Gamma,\ \hat{u}(\ell) = u(\ell).$$

Since k_3 is concave, (5.1) implies a normalization of k_3 in such a way that

(5.2) $$k_3(u^3(a^1)) \leq u^3(a^1),\ k_3(0) = 0,\ k_3(u^3(a^i)) \geq u^3(a^i) \text{ if } i \in \{2,3\},$$
$$k_3(u^3(a^4)) \leq u^3(a^4).$$

Suppose $\phi_3(K^3(\Gamma)) > u^3(\ell)$. Let S be the disposable hull of $\text{conv}\{(0,0,0),\ u(\ell),\ \phi(K^3(\Gamma))\}$. Then, by (5.2), $S \subset S_\Gamma$, and, by (5.1), $S \subset S_{K^3(\Gamma)}$. Hence, by applying IIA twice, we obtain $\phi(S,0) = \phi(\Gamma) = \phi(K^3(\Gamma))$, a contradiction since we assumed $\phi_3(K^3(\Gamma)) > u^3(\ell) = \phi_3(\Gamma)$. So $\phi_3(K^3(\Gamma)) \leq u^3(\ell)$ which, by (5.2), implies that $u^3(\ell) \geq u^3(\hat{\ell})$, and then also $\hat{u}^3(\ell) \geq \hat{u}^3(\ell)$.

(ii) The proof is almost similar to the proof of (i), except that, instead of (5.2), we now have, after applying normalization (5.1),

(5.3) $$k_3(u^3(a^1)) \leq u^3(a^1),\ k_3(0) = 0,\ k_3(u^3(a^i)) \geq u^3(a^i) \text{ if } i \in \{2,3\}.$$

Now, the assumption $\phi_3(K^3(\Gamma)) < u^3(\ell)$ will lead to a contradiction, and by (5.3), we then have $u^3(\ell) \leq u^3(\hat{\ell})$, $\hat{u}^3(\ell) \leq \hat{u}^3(\hat{\ell})$. □

Following Roth and Rothblum [18], we might say that in the first case the solution is *favorably* u-*supported for player* 3: in that case, it is a disadvantage for player 3 to be more risk averse; whereas in the second case we might say that the solution is *unfavorably u-supported for player* 3: in this case, it is no disadvantage for player 3 to be more risk averse.

EXAMPLE 4. Let $\phi^H : BS^3 \to \mathbb{R}^3$ be the (unique symmetric) 3-person IIA-solution corresponding to the weighted ordered partition $H = \langle\{1,2,3\},(\frac{1}{3},\frac{1}{3},\frac{1}{3})\rangle$. Let $\Gamma = \langle A,\bar{a},u^1,u^2,u^3\rangle \in BS^3$ such that $A = \{\bar{a},a^1,a^2,a^3,a^4\}$, $(d_\Gamma =)\ u(\bar{a}) = (0,0,0)$, $u(a^1) = (1,0,-\frac{1}{2})$, $u(a^2) = (0,1,-\frac{1}{2})$, $u(a^3) = (0,0,1)$, $u(a^4) = (\frac{2}{5},\frac{2}{5},\frac{1}{5})$. Let $k_3 \in C_3(\Gamma)$ such that $k_3(\lambda) = 4\lambda$ for all $\lambda \in (-\infty,0]$, $k_3(\lambda) = \lambda$ for all $\lambda \in [0,\frac{1}{5}]$, $k_3(\lambda) = \frac{3}{4}\lambda + \frac{1}{20}$ for all $\lambda \in [\frac{1}{5},\infty)$.

The following facts can be verified by elementary calculations.

$\phi(\Gamma) = (\frac{1}{3},\frac{1}{3},\frac{1}{3}) = u(\ell)$ where $\ell = [\frac{5}{6}a^4;\ \frac{1}{6}a^3]$, and

$O_{-3}(\Gamma,\phi^H) = \mathrm{conv}\{(0,0),(\frac{1}{3},\frac{1}{3}),(\frac{4}{9},0),(0,\frac{4}{9})\}$.

And $\phi(K^3(\Gamma)) = (\frac{16}{45},\frac{16}{45},\frac{12}{45})$, which corresponds to the lottery $\hat{\ell} = [\frac{8}{9}a^4;\ \frac{1}{9}a^3]$, whereas

$O_{-3}(K^3(\Gamma),\phi^H) = \mathrm{conv}\{(0,0),(\frac{16}{45},\frac{16}{46}),(\frac{16}{45},0),(0,\frac{16}{45})\}$.

In this case, the lottery ℓ is strictly preferred by both player 3 and his more risk averse substitute to the lottery $\hat{\ell}$, but the opportunity set $O_{-3}(\Gamma,\phi^H)$ is not a subset of the opportunity set $O_{-3}(K^3(\Gamma),\phi^H)$. This shows that theorem 2 cannot be extended to games with risky Pareto optimal outcomes.

6. CONCLUDING REMARKS

We summarize the main results of this paper in a few diagrams. The first diagram holds for bargaining maps (i.e., satisfying PO and IR) on BSC^n. Here, A denotes the property described in theorem 2(ii).

$$\begin{tikzcd}
A \arrow[d, Leftrightarrow] & RS \arrow[d, Rightarrow] & \\
RPO \arrow[d, Rightarrow, "n=2"'] & \& \arrow[d, Rightarrow] & EC \\
RS & IEUT &
\end{tikzcd}$$

The second diagram holds for bargaining solutions (i.e., satisfying PO, IR and IEUT) on BSC^n.

$$\begin{tikzcd}
 & & A \arrow[d, Leftrightarrow] & \\
IM \arrow[r, Rightarrow] & RS \arrow[r, Rightarrow] \arrow[d, Rightarrow] & RPO \arrow[r, Rightarrow, "n=2"'] & RS \\
 & EC & IIA \arrow[l, Rightarrow] \arrow[u, Rightarrow] &
\end{tikzcd}$$

For games with risky Pareto optimal outcomes, not many regularities can be obtained with respect to risk behaviour of bargaining solutions. If, however, we want to look for partial results, then the examples of section 5 indicate that property A (i.e. the property where lotteries are compared instead of utilities) is the proper one to consider. For bargaining maps defined on games with only riskless Pareto optimal outcomes, this property is equivalent to RPO. The RPO property however is, like the RS property for n=2, mathematically more convenient to handle.

Finally, we remark that relations between risk properties and some other properties are investigated in Tijs and Peters [20].

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